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1996 J. Phys. A: Math. Gen. 29 5735

(http://iopscience.iop.org/0305-4470/29/18/008)

J. Phys. A: Math. Gen. **29** (1996) 5735–5743. Printed in the UK

Fusion procedure for the Z_n Belavin model with open boundary conditions

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Received 6 December 1995, in final form 4 June 1996

Abstract. We have recently used unitarity and cross-unitarity properties of an Z_n symmetric R-matrix to construct the transfer matrix $t(u)$ for an N-site open spin chain. Here we give a fusion procedure for such a chain, and we prove that the fused transfer matrix $\tilde{t}(u)$ is commutative with the original transfer matrix $t(u)$.

1. Introduction

It is well known that the integrability of two-dimensional lattice models is a consequence of the Yang–Baxter equation [1, 2], which is usually written as

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{1}$$

where the R-matrix can be regarded as the Boltzmann weight for the vertex models in two-dimensional statistical mechanics; as usual, $R_{12}(u)$, $R_{13}(u)$ and $R_{23}(u)$ act in $C^n \otimes C^n \otimes C^n$, with $R_{12}(u) = R(u) \otimes 1$, $R_{23}(u) = 1 \otimes R(u)$, etc. It has been pointed out that the solution of the Yang–Baxter equation can be related to some algebraic theories such as quantum group and Sklyanin algebra. In the study of the Yang–Baxter equation the so-called fusion procedure was developed to generate new integrable models corresponding to the group invariant solutions of the Yang–Baxter equation from a known model.

Recently, much more attention has been paid to integrable systems with open boundary conditions, which was initiated by Cherednik [3] and Sklyanin [4]. They introduced a systematic approach to handle the finite-size systems which involve the so-called reflection equation:

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v) \tag{2}$$

$$\begin{aligned} &R_{12}(-u+v)K_1^+(u)^{t_1}R_{21}(-u-v-2\eta)K_2^+(v)^{t_2} \\ &\qquad = K_2^+(v)^{t_2}R_{12}(-u-v-2\eta)K_1^+(u)^{t_1}R_{21}(-u+v) \end{aligned} \tag{3}$$

$K_1^{\pm}(u)$ and $K_2^{\pm}(u)$ are boundary matrices acting in $C^n \otimes 1$ and $1 \otimes C^n$, respectively, which determine uniquely the boundary terms in Hamiltonian, and η is a constant characterizing the R-matrix. Using this approach, Sklyanin [4] solved the open spin-$\frac{1}{2}$ XXZ model with general boundary conditions by generalizing the quantum inverse scattering method (QISM). The transfer matrix with particular boundary conditions is $U_q[sl(2)]$ invariant [5]. The R-matrix is assumed to satisfy both P- and T-symmetry as well as unitarity and cross-unitarity properties. Because only a few models satisfy these properties, Mezincescu and Nepomechie

0305-4470/96/185735+09$19.50

[6] generalized Sklyanin's approach of constructing integrable open chains to the case where the R-matrix is only PT-invariant. So, all of the trigonometric R-matrices listed by Jimbo [7] and Bazhanov [8] are of this type, and other models such as supersymmetric t–j models [9] and the Perk–Schultz model [10] also satisfy these properties [11]. Correspondingly, the reflection equations are extended to the modified reflection equations. The boundary matrices $K^{\pm}(u)$ are solutions of the modified reflection equations.

However, for the $Z_n \times Z_n$ Belavin model [12, 13], where the R-matrix does not have the property of PT-symmetry, the unitarity and cross-unitarity relations take a different form from Sklyanin's formalism and it's generalization. We have recently shown [14] that one can only use unitarity and cross-unitarity properties of a Z_n symmetric R-matrix to construct the transfer matrix for an open chain; this transfer matrix forms a one-parameter commutative family which ensures the integrability of the system under consideration. The reflection equations take the following form:

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v) \tag{4}$$

$$\begin{aligned} &R_{12}(-u+v)K_1^+(u)R_{21}(-u-v-nw)K_2^+(v) \\ &\qquad = K_2^+(v)R_{12}(-u-v-nw)K_1^+(u)R_{21}(-u+v). \end{aligned} \tag{5}$$

Obviously, there is an isomorphism between $K^+(u)$ and $K^-(u)$,

$$\phi : K^-(u) \to K^+(u) = K^-(-u - \tfrac{1}{2}nw). \tag{6}$$

This relation means that given a solution $K^-(u)$ of (4), one can also find a solution $K^+(u)$ of (5).

Mezincescu and Nepomechie have formulated a fusion procedure for boundary $K^{\pm}$ matrices corresponding to an R-matrix which is PT-invariant; these results can be used to construct integrable open chains of higher spin [15, 16]. By using this approach, Yu-kui Zhou has studied the fused six-vertex models with open boundary conditions. The central charge and conformal weights of underlying conformal field theory are extracted from finite-size corrections of the fused transfer matrices for low-lying excitations [17]. He has also studied the functional relations of the transfer matrices of fusion hierachies for the eight-vertex model with open boundary conditions [18]. For RSOS models, the fusion procedure for the reflection matrix $K^{\pm}$ of the reflection equations has been presented in [19]. It is known that the Z_n Belavin model reduces to Baxter's eight-vertex model in the case $n = 2$. By taking a limit of the Z_n symmetric Belavin R-matrix, one can obtain a trigonometric R-matrix from which we can obtain the quantum group $sl_q(n)$ [20]. When $n = 2$, this R-matrix is just a six-vertex R-matrix. In this paper, we will formulate a fusion procedure for the Z_n Belavin model with open boundary conditions. The fused transfer matrix $\tilde{t}(u)$ can be proved to be commutative with the original transfer matrix $t(u)$.

The outline of this paper is as follows. In section 2, we construct the Z_n Belavin vertex R-matrix. In section 3, we carry out the fusion procedure for R matrices. In section 4, we formulate the fusion procedure for reflection $K^{\pm}$ matrices. Section 5 is devoted to the fused transfer matrix with open boundary conditions. A proof that the fused transfer matrix commutes with the original transfer matrix is given. Section 6 contains a brief summary and some discussions.

2. Description of the model

First, we construct the matrix of vertex weights of the Belavin $Z_n \otimes Z_n$ symmetric model. Let g, h, I_α be $n \times n$ matrices with elements $g_{jk} = \omega^j \delta_{jk}$, $h_{jk} = \delta_{j+1,k}$, $I_\alpha = I_{(\alpha_1,\alpha_2)} = g^{\alpha_2} h^{\alpha_1}$,

$I_0 = I$, where $i, j, k, \alpha_1, \alpha_2 \in Z_n$, $\omega = e^{2\pi i/n}$. Define $I_\alpha^{(j)} = I \otimes \cdots \otimes I_\alpha \otimes I \otimes \cdots \otimes I$, I_α is at jth space.

$$W_\alpha(u) \equiv \frac{\theta\begin{bmatrix} \frac{1}{2}+\frac{\alpha_1}{n} \\ \frac{1}{2}+\frac{\alpha_2}{n} \end{bmatrix}(u+\frac{w}{n},\tau)}{\theta\begin{bmatrix} \frac{1}{2}+\frac{\alpha_1}{n} \\ \frac{1}{2}+\frac{\alpha_2}{n} \end{bmatrix}(\frac{w}{n},\tau)} = \frac{\sigma_\alpha(u+\frac{w}{n})}{\sigma_\alpha(\frac{w}{n})} \tag{7}$$

where

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(u,\tau) = \sum_{m\in Z} \exp\{i\pi(m+a)^2\tau + 2\pi i(m+a)(u+b)\}. \tag{8}$$

The Z_n symmetric Belavin R-matrix takes the form

$$R_{jk}(u) = \frac{1}{n}\sum_{\alpha\in Z_n^2} W_\alpha(u) I_\alpha^{(j)} (I_\alpha^{-1})^{(k)}. \tag{9}$$

This satisfies the Yang–Baxter equation (1). The elements of the R-matrix have been written out explicitly by Richey and Tracy [13].

$$R_{ij}^{kl}(u) = \begin{cases} \dfrac{h(u)\theta^{(i-j)}(u+w)}{(\theta^{(i-k)}(w)\theta^{(k-j)}(u)} & \text{for } i+j=k+l \bmod n \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where

$$h(u) = \prod_{j=0}^{n-1}\theta^{(j)}(u) \Big/ \prod_{j=1}^{n-1}\theta^{(j)}(0) \qquad \theta^{(i)}(u) = \begin{bmatrix} \frac{1}{2}-\frac{i}{n} \\ \frac{1}{2} \end{bmatrix}(z, n\tau).$$

The R-matrix of the Z_n Belavin model satisfies the following initial condition, unitarity and cross-unitarity properties:

$$R_{12}(0) = P_{12} \tag{11}$$

$$R_{12}(u)R_{21}(-u) = \rho(u)\cdot \mathrm{id} \tag{12}$$

$$R_{12}^{t_1}(u)R_{21}^{t_2}(-u-nw) = \tilde{\rho}(u)\cdot \mathrm{id} \tag{13}$$

where

$$\begin{aligned} \rho(u) &= \frac{\sigma(u+w)\sigma(-u+w)}{\sigma^2(w)} \\ \tilde{\rho}(u) &= \frac{\sigma(u)\sigma(-u-nw)}{\sigma^2(w)} \\ \sigma(u) &\equiv \sigma_0(u). \end{aligned} \tag{14}$$

P_{12} is the permutation operator.

Define

$$W_\alpha(u,\xi) = \frac{\sigma_\alpha(u+\xi)}{\sigma_\alpha(\xi)} \tag{15}$$

$$K(u,\xi) = \frac{1}{n}\sum_{\alpha\in Z_n^2} W_{2\alpha}(u,\xi)\omega^{2\alpha_1\alpha_2} I_{2\alpha} \tag{16}$$

where ξ is an arbitrary parameter. We have proved in [14, 21, 22] that $K^-(u,\xi_-) = K(u,\xi_-)K(0)$ is a solution of the reflection equation (4), and, correspondingly, $K^+(u,\xi_+) = K(-u-1/2nw,\xi_+)K(0)$ is a solution of the reflection equation (5).

3. Review of the fusion procedure of the R-matrix

We describe first the fusion procedure of the Z_n symmetric Belavin vertex model. From (10) we have [23, 24],

$$R_{12}(-w) = A^- P_{12}^-, R_{12}(w) = P_{12}^+ A^+ \tag{17}$$

$$P_{12}^{\pm} \equiv \tfrac{1}{2}(1 \pm P_{12}). \tag{18}$$

One should notice the order of $A^{\pm}$ and $P^{\pm}$. We can prove easily that $\det(A^{\pm}) \neq 0$ as $\tau \to \mathrm{i}\infty$ or $w \to 0$. This leads to $\det(A^{\pm}) \neq 0$ for almost all w. Hence matrices $A^{\pm}$ are invertible for almost all w. We refer the readers to [23] for the details of the matrices $A^{\pm}$. Note that these relations are essential for establishing the fusion procedure.

We define the fused R-matrix as

$$R_{\langle 12\rangle 3}(u) = P_{12}^+ R_{13}(u) R_{23}(u+w) P_{12}^+. \tag{19}$$

By using the original Yang–Baxter equation (1) several times, we can prove that the fused R-matrix satisfies the following generalized Yang–Baxter equation:

$$R_{\langle 12\rangle 3}(u-v) R_{\langle 12\rangle 4}(u) R_{34}(v) = R_{34}(v) R_{\langle 12\rangle 4}(u) R_{\langle 12\rangle 3}(u-v). \tag{20}$$

Similiarly, we can also define another type of fused R-matrix

$$R_{3\langle 12\rangle}(u) = P_{12}^+ R_{32}(u-w) R_{31}(u) P_{12}^+ \tag{21}$$

satisfying a generalized Yang–Baxter equation. Actually we can construct a fusion hierarchy of the Z_n Belavin vertex R-matrix satisfying the generalized Yang–Baxter equations:

$$R_{s_1 s_2}(u-v) R_{s_1 s_3}(u) R_{s_2 s_3}(v) = R_{s_2 s_3}(v) R_{s_1 s_3}(u) R_{s_1 s_2}(u-v). \tag{22}$$

It is convenient to introduce the following notations

$$R'_{\langle 12\rangle 3}(u) = R_{\langle 21\rangle 3}(u-w) \tag{23}$$

$$R'_{3\langle 12\rangle}(u) = R_{3\langle 21\rangle}(u+w). \tag{24}$$

Next, we will find the unitarity and cross-unitarity relations of the fused R-matrices which are essential in the rest of our paper. We list here two forms of the unitarity relations of the fused R-matrices:

$$R_{\langle 12\rangle 3}(u) R_{3\langle 12\rangle}(-u) = \rho(u)\rho(u+w) P_{12}^+ = f(u) P_{12}^+ \tag{25}$$

$$R'_{3\langle 12\rangle}(u) R'_{\langle 12\rangle 3}(-u) = \rho(u)\rho(u+w) P_{21}^+ = f(u) P_{21}^+. \tag{26}$$

which follows directly from the unitarity relation (12).

As for the cross-unitarity relations of the fused R-matrices, we find:

$$\begin{aligned} R'^{t_3}_{3\langle 12\rangle}(-u-nw) R'^{t_3}_{\langle 12\rangle 3}(u) &= \frac{1}{n^4} \sum_{\alpha\beta\gamma\delta} W_\alpha(-u-nw) W_\delta(u) \times W_\beta(-u-nw+w) W_\gamma(u-w) \\ I_\alpha^{-1} I_\delta \otimes I_\beta^{-1} I_\gamma \otimes I_\beta^t (I_\alpha^t I_\delta^{-1t}) I_\gamma^{-1t} &= \tilde{\rho}(u)\tilde{\rho}(u-w) \cdot P_{12}^+ = \tilde{f}(u) \cdot P_{12}^+. \end{aligned} \tag{27}$$

Here we have used the cross-unitarity relation (13) of the original R-matrix. We can similiary find another form of cross-unitarity relation for the fused R-matrix:

$$R^{t_1 t_2}_{\langle 12\rangle 3}(-u-nw) R^{t_1 t_2}_{3\langle 12\rangle}(u) = \tilde{f}(u) \cdot P_{12}^+. \tag{28}$$

4. Fusion procedure for the reflection K-matrices

In this section, we will formulate a fusion procedure for reflection K-matrices. As mentioned above, $K^{\pm}$ matrices determine the non-trivial boundary terms in the Hamiltonian. So the fused K-matrices determine the boundary terms of the higher spin chains.

We will employ the same method used for the R-matrix for the reflection K-matrix. Taking $v = u + w$, with the help of equations (17), the reflection equation (4) becomes

$$A^- P_{12}^- K_1^-(u) R_{21}(2u+w) K_2^-(u+w) = K_2^-(u+w) R_{12}(2u+w) K_1^-(u) A_- P_{21}^-. \tag{29}$$

This leads to the relation

$$P_{12}^- K_1^-(u) R_{21}(2u+w) K_2^-(u+w) P_{21}^+ = 0. \tag{30}$$

Hence, we can define the fused K^- matrix as

$$K_{\langle 12\rangle}^-(u) = P_{12}^+ K_1^-(u) R_{21}(2u+w) K_2^-(u+w) P_{21}^+. \tag{31}$$

One can prove that the fused reflection K^- matrix satisfies the following generalized reflection equation:

$$R_{3\langle 12\rangle}(u-v) K_3^-(u) R_{\langle 12\rangle 3}(u+v) K_{\langle 12\rangle}^-(v) = K_{\langle 12\rangle}^-(v) R'_{3\langle 12\rangle}(u+v) K_3^-(u) R'_{\langle 12\rangle 3}(u-v) \tag{32}$$

as follows from the Yang–Baxter equation (1) and the reflection equation (4). Similiarly, following the same strategy employed for K^-, we can also find the fusion procedure for K^+. Setting $v = u - w$, the fused K^+ matrix takes the form

$$K_{\langle 12\rangle}^+(u) = P_{12}^+ K_1^+(u) R_{21}(-2u+w-nw) K_2^+(u-w) P_{12}^+. \tag{33}$$

By using the Yang–Baxter equation (1) and reflection equation (5), we are led to the following generalized reflection equation for the fused reflection K^+ matrix:

$$\begin{aligned} R_{3\langle 12\rangle}(-u+w) K_3^+(u) R_{\langle 12\rangle 3}(-u-v-nw) K_{\langle 12\rangle}^+ \\ = K_{\langle 12\rangle}^+(v) R'_{3\langle 12\rangle}(-u-v-nw) K_3^+(u) R'_{\langle 12\rangle 3}(-u+v). \end{aligned} \tag{34}$$

Comparing the two generalized reflection equations (32) and (34), one can also find an isomorphism between fused K^+ and K^- matrices. This means that equation (6) still holds for fused K-matrices.

Here, we will present another form of the generalized reflection equation (32) for the fused K^- matrix:

$$\begin{aligned} R'_{3\langle 12\rangle}(u-v) K_3^-(u) R'_{\langle 12\rangle 3}(u+v) K_{\langle 21\rangle}(v-w) \\ = K_{\langle 21\rangle}^-(v-w) R_{3\langle 12\rangle}(u+v) K_3^-(u) R_{\langle 12\rangle 3}(u-v). \end{aligned} \tag{35}$$

This can be obtained from equation (32) by using the transformation: *space 1* $\leftrightarrow$ *space 2*; $v \to v - w$.

5. The fused transfer matrix with open boundary conditions

It is well known that in the framework of the quantum inverse scattering method [25] the Yang–Baxter equation takes the form

$$R_{12}(u-v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u-v) \tag{36}$$

where $T(u)$ is the standard row-to-row monodromy matrix. Using the same fusion procedure for the R-matrix, one can obtain the fusion procedure for the monodromy matrix

$$T_{\langle 12\rangle}(u) = P_{12}^+ T_1(u) T_2(u+w) P_{12}^+ \tag{37}$$

satisfying the relation

$$R_{3\langle 12\rangle}(u-v)T_2(u)T_{\langle 12\rangle}(v) = T_{\langle 12\rangle}(v)T_3(u)R_{3\langle 12\rangle}(u-v). \tag{38}$$

Denoting $T^{-1}(-u)$ by $\hat{T}(u)$ as used by Mezincescu and Nepomechie [15], we can transform the Yang–Baxter equation:

$$R_{12}(v-u)\hat{T}_2(v)\hat{T}_1(u) = \hat{T}_1(u)\hat{T}_2(v)R_{12}(v-u). \tag{39}$$

Repeating the same fusion procedure, we find

$$\hat{T}_{\langle 12\rangle}(u) = P_{12}^{+}\hat{T}_2(u-w)\hat{T}_1(u)P_{12}^{+} \tag{40}$$

which satisfies the relation

$$R_{\langle 12\rangle 3}(v-u)\hat{T}_3(v)\hat{T}_{\langle 12\rangle}(u) = \hat{T}_{\langle 12\rangle}(u)\hat{T}_3(v)R_{\langle 12\rangle 3}(v-u). \tag{41}$$

In the case of open boundary conditions, we define the double-row monodromy matrix as

$$\mathcal{T}(u) = T(u)K^{-}(u)\hat{T}(u) \tag{42}$$

satisfying the same reflection equation as that which K^{-} satisfied:

$$R_{12}(u-v)\mathcal{T}_1(u)R_{21}(u+v)\mathcal{T}_2(v) = \mathcal{T}_2(v)R_{12}(u+v)\mathcal{T}_1(u)R_{21}(u-v). \tag{43}$$

So we can similiarly find a fusion procedure for the double-row monodromy matrix:

$$\mathcal{T}_{\langle 12\rangle}(u) = P_{12}^{+}\mathcal{T}_1(u)R_{21}(2u+w)\mathcal{T}_2(u+w)P_{21}^{+}. \tag{44}$$

One can prove that the fused double-row monodromy matrix satisfies the same relations as that of $K_{\langle 12\rangle}^{-}(u)$:

$$\begin{aligned}\mathcal{T}_{\langle 21\rangle}(v-w)&R_{3\langle 12\rangle}(u+v)\mathcal{T}_3(u)R_{\langle 12\rangle 3}(u-v)\\ &= R'_{3\langle 12\rangle}(u-v)\mathcal{T}_3(u)R'_{\langle 12\rangle 3}(u+v)\mathcal{T}_{\langle 21\rangle -}(v-w).\end{aligned} \tag{45}$$

Here, we will introduce another definition for the fused double-row monodromy matrix:

$$\mathcal{T}_{\langle 12\rangle}(u) = T_{\langle 12\rangle}(u)K_{\langle 12\rangle}^{-}(u)\hat{T}_{\langle 21\rangle}(u+w). \tag{46}$$

The two definitions (44) and (46) can be proved to be equivalent to each other:

$$\begin{aligned}\mathcal{T}_{\langle 12\rangle}(u) = T_{\langle 12\rangle}(u)&K_{\langle 12\rangle}^{-}(u)\hat{T}_{\langle 21\rangle}(u+w)\\ &= T_1(u)K_1^{-}(u)T_2(u+w)R_{21}(2u+w)\hat{T}_1(u)K_2^{-}(u+w)\hat{T}_1(u)\hat{T}_2(u+w)P_{21}^{+}\\ &= T_1(u)K_1^{-}(u)\hat{T}_1(u)R_{21}(2u+w)T_2(u+w)K_2^{-}(u+w)\hat{T}_2(u+w)P_{21}^{+}\\ &= P_{12}^{+}\mathcal{T}(u)R_{21}(2u+w)\mathcal{T}_2(u+w)P_{21}^{+}.\end{aligned} \tag{47}$$

We know that the original transfer matrix with open boundary conditions is defined as

$$t(u) = \mathrm{tr}K^{+}(u)\mathcal{T}(u). \tag{48}$$

By using the unitarity and cross-unitarity properties of the Z_n symmetric R-matrix, we have proved that this transfer matrix constitutes a one-parameter commutative family. For the case of fusion, we define the fused transfer matrix with open boundary conditions as

$$\tilde{t}(u) = \mathrm{tr}_{12}\, K_{\langle 12\rangle}^{+}(u)\mathcal{T}_{\langle 21\rangle}(u-w). \tag{49}$$

Next, we will prove that the fused transfer matrix commutes with the original transfer matrix $t(u)$, which means that $\tilde{t}(u)$ also forms a commuting family.

We first insert the unitarity and cross-unitarity relations (26) and (27) into the following relations:

$$
\begin{aligned}
t(u)\tilde{t}(v) &= \mathrm{tr}_{123}\, K_3^{+t_3}(u) K^{+}_{\langle 12\rangle}(v) \mathcal{T}_3^{t_3}(u) \mathcal{T}_{\langle 21\rangle}(v) \\
&= \frac{1}{\tilde{f}(u_+)}\, \mathrm{tr}_{123}\, K_3^{+t_3}(u) K^{+}_{\langle 12\rangle}(v) \\
&\quad \times \{R'^{t_3}_{3\langle 12\rangle}(-u_+ - nw) R'^{t_3}_{\langle 12\rangle 3}(u_+)\} \mathcal{T}_3^{t_3}(u) \mathcal{T}_{\langle 21\rangle}(v-w) \\
&= \frac{1}{\tilde{f}(u_+) f(u_-)}\, \mathrm{tr}_{123} \{K_3^{+t_3}(u) R'^{t_3}_{3\langle 12\rangle}(-u_+ - nw) K^{+t_1t_2}_{\langle 12\rangle}(v)\} \\
&\quad \times \{\mathcal{T}_3(u) R'_{\langle 12\rangle 3}(u_+) \mathcal{T}_{\langle 21\rangle}(v-w)\}^{t_1t_2t_3} \{R'^{t_1t_2t_3}_{3\langle 12\rangle}(u_-) R'^{t_1t_2t_3}_{\langle 12\rangle 3}(-u_-)\} \\
&= \frac{1}{\tilde{f}(u_+) f(u_-)}\, \mathrm{tr}_{123} \{R'_{\langle 12\rangle 3}(-u_-)^{t_1t_2t_3} K_3^{+t_3}(u) R'^{t_3}_{3\langle 12\rangle}(-u_+ - nw) \\
&\quad \times K^{+t_1t_2}_{\langle 12\rangle}(v)\}^{t_1t_2t_3} \{R'_{3\langle 12\rangle}(u_-) \mathcal{T}_3(u) R'_{\langle 12\rangle 3}(u_+) \mathcal{T}_{\langle 21\rangle}(v-w)\}.
\end{aligned}
$$

Here notations $u_\pm = u \pm v$ have been used. Then, applying the transposition $t_1t_2t_3$ and using the reflection equations (34) and (35):

$$
\begin{aligned}
\cdots &= \frac{1}{\tilde{f}(u_+) f(u_-)}\, \mathrm{tr}_{123} \{R_{3\langle 12\rangle}(-u_-) K_3^{+}(u) R_{\langle 12\rangle 3}(-u_+ - nw) K^{+}_{\langle 12\rangle}(v)\} \\
&\quad \times \{\mathcal{T}_{\langle 21\rangle}(v-w) R_{3\langle 12\rangle}(u_+) \mathcal{T}_3(u) R_{\langle 12\rangle 3}(u_-)\} \\
&= \frac{1}{\tilde{f}(u_+)}\, \mathrm{tr}_{123} \{K_3^{+}(u) K^{+t_1t_2}_{\langle 12\rangle}(v) R^{t_1t_2}_{\langle 12\rangle 3}(-u_+ - nw)\} \\
&\quad \times \{R^{t_1t_2}_{3\langle 12\rangle}(u_+) \mathcal{T}^{t_1t_2}_{\langle 21\rangle}(v-w) \mathcal{T}_3(u)\} \\
&= \mathrm{tr}_{12}\, K^{+t_1t_2}_{\langle 12\rangle}(v) \mathcal{T}^{t_1t_2}_{\langle 21\rangle}(v-w)\, \mathrm{tr}_3\, K_3^{+}(u) \mathcal{T}_3(u) \\
&= \tilde{t}(v) t(u).
\end{aligned} \tag{50}
$$

Here the unitarity relation (25) and the cross-unitarity relation (28) have been applied. Thus we have proved that the fused transfer matrix with open boundary conditions commutes with the original transfer matrix with open boundary conditions.

6. Summary and discussions

In conclusion, we have formulated a fusion procedure for the Z_n Belavin vertex R-matrix model with open boundary conditions. We have obtained the fused reflection K-matrices, double-row monodromy matrix and transfer matrix with open boundary conditions. We have also proved that the fused transfer matrix is commutative with the original transfer matrix so that the fused transfer matrix also constitutes a commuting family.

In this paper, we have given a fusion procedure for the Z_n Belavin model. Following the same method, one can generalize our formalism to higher levels. Defining the projector

$$
Y_p^+ = \frac{1}{p!}(P_{1,p} + \cdots + P_{p-1,p} + I) \cdots (P_{12} + I) \tag{51}
$$

the fused R-matrix can be given by [16]

$$
\begin{aligned}
R_{(p,q)}(u) &= Y_q^+ (R_{(p)q}(u - qw + w) \cdots R_{(p)2}(u-w) R_{(p)1}(u)) Y_q^+ \\
R_{(p)j}(u) &= Y_p^+ R_{1j}(u) R_{2j}(u+w) \cdots R_{pj}(u + pw - w) Y_p^+.
\end{aligned} \tag{52}
$$

Following the method presented in this paper, we assume the level q fused K^- matrix takes the form

$$\begin{aligned}K^-_{(q)}(u) = Y^+_q[K^-_1(u)][R_{21}(2u+w)K^-_2(u+w)]\cdots\\ \cdots[R_{q,1}(2u+(q-1)w)R_{q-1,1}(2u+(q-2)w)\cdots\\ \cdots R_{21}(2u+w)K^-_q(u+(q-1)w)]Y^+_q.\end{aligned}\tag{53}$$

Similarly, the level q fused K^+ matrix could be defined as

$$\begin{aligned}K^+_{(q)}(u) = Y^+_q[K^+_1(u)][R_{21}(-2u+w-nw)K^+_2(u-w)]\cdots\\ \cdots[R_{q,1}(-2u+(q-1)w-nw)\cdots\\ \cdots R_{21}(-2u+w-nw)K^+_q(u-(q-1)w)]Y^+_q\end{aligned}\tag{54}$$

which satisfies the fused reflection equations.

Similar to the fusion of the R-matrix, the high-level fused row-to-row monodromy matrices $T(u)$ and $\hat{T}(u)$ are defined as

$$\begin{aligned}T_{(p,q)}(u) = Y^+_q T_{1(p)}(u)T_{2(p)}(u+w)\cdots T_{q(p)}(u+(q-1)w)Y^+_q\\ \hat{T}_{p,q}(u) = Y^+_q\hat{T}_{1(p)}(u-qw)\hat{T}_{2(p)}(u-(q-1)w)\cdots\hat{T}_{q(p)}(u-w)Y^+_q\end{aligned}\tag{55}$$

where

$$\begin{aligned}T_{j(p)}(u) = R^{(1,p)}_{j1}(u)R^{(1,p)}_{j2}(u)\cdots R^{(1,p)}_{jN}(u)\\ \hat{T}_{j(p)}(u) = R^{(1,p)}_{jN}(-u)^{-1}\cdots R^{(1,p)}_{j1}(-u)^{-1}.\end{aligned}\tag{56}$$

Define the generalized fused transfer matrix with open boundary conditions as

$$t^{(p,q)}(u) = \mathrm{tr}\, K^+_{(q)}(u)T_{(p,q)}(u)K^-_{(q)}(u)\hat{T}_{(p,q)}(u).\tag{57}$$

The fused Yang–Baxter equation and the fused reflection equations guarantee the following commuting families:

$$[t^{(p,q)}(u), t^{(p,b)}(v)] = 0.\tag{58}$$

It is easy to see that $t^{(1,2)}(u) = \tilde{t}(u)$, $t^{(1,1)}(u) = t(u)$ which have already been defined. So, we have constructed a fusion hierarchy and one can extract the following functional Bethe ansatz equations,

$$t^{(p,q)}(u)t^{(p,q)}(u-w) = t^{(p,q+1)}(u)t^{(p,q-1)}(u-w) + t^{(p+1,q)}(u)t^{(p-1,q)}(u-w)\tag{59}$$

satisfying the $su(n)$ fusion hierarchy. These functional equations can be converted into the so-called thermodynamic Bethe-ansatz-like equations [26] and can be solved analytically for finite-size scaling spectra, central charges and conformal weights [27]. For the eight-vertex model, which has been studied in [18], for the Z_n Belavin model, this needs further studies.

We know that an important development in exactly solvable models with open boundary conditions is the boundary cross-unitarity relation proprosed by Ghoshal and Zamolodchikov [28, 29]. The boundary cross-unitarity relations have been obtained for six-vertex, eight-vertex and O(n) sigma models; for the general R-matrix with $n > 2$, the cross-unitarity relations have not been obtained before. By using the fusion procedure for the Z_n Belavin model similar to the method in this paper, the boundary cross-unitarity relation has recently been obtained [30] and it takes the form

$$[K^-(u)^*]^k_j = R(2u+nw)^{lj}_{kl'}K^-(-u-nw)^{l'}_l\tag{60}$$

where a factor has been omitted,

$$\begin{aligned}K^-(u)^* = C^{-1}Y^-_{(n-1)}K^-_n(u+(n-1)w)R_{n-1,n}(2u+(2n-3)w)\cdots\\ \cdots K^-_2(u+w)Y^-_{(n-1)}C\end{aligned}\tag{61}$$

where C is a matrix related to the R-matrix. We know that by using the boundary unitarity and boundary cross-unitarity relations one could obtain the boundary free energy and surface critical exponents etc which are of interest to physicists. For the eight-vertex model, this has already be studied by Batchelor *et al* [31]. It is also important to obtain the difference equations for the correlation functions of the Z_n Belavin model with open boundary conditions following the method in [32].

For the Z_n Belavin model with periodic boundary conditions, Hasegawa has given a Macdonald-type operator which is equivalent to Ruijsenaars' operators by using the fusion procedure [33]. For open boundary conditions, using the commuting families obtained in this paper, one could also find analogous Macdonald-type operators.

Acknowledgments

One of the authors, H Fan, would like to thank Dr Z Yang, Dr R Yue and Professor Y Zhou for useful discussions, he would also like to thank Professor P Wang for encouragement. This work was partly supported by the Natural Science Foundation of China.

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